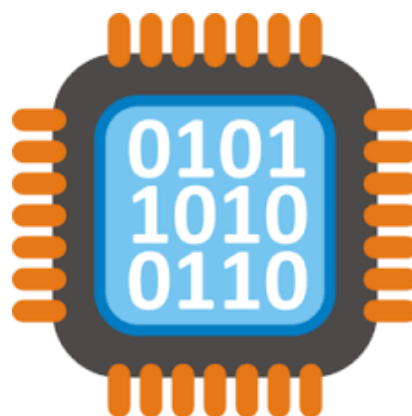

THE PYTHAGOREAN THEOREM WITH RADICALS

The last time we studied the Pythagorean Theorem we may have used our calculator to round off square roots that didn't come out whole numbers. But rounding off square roots is not always a good idea in the real world.

After we've done some applied mathematics, we may give the results to an engineer. Suppose we've rounded to the hundredths place (good enough for business), but the engineer is designing the Core 9 Quad microprocessor, and she needs all the digits of the calculations out to the billionths place. We haven't done her any good at all by rounding off for her. The safest thing to do is to give her the exact results and let her round off according to the requirements of her project.



We're now going to work with the Pythagorean Theorem again, but this time we're going to leave our answers exact. To this end, we need to review the key concept regarding square roots.

❑ ***AN IMPORTANT FACT ABOUT SQUARE ROOTS***

If we square the square root of “something” (that’s not negative), we get the “something”:

$$(\sqrt{25})^2 = 5^2 = 25$$

$$(\sqrt{34})^2 \approx (5.830951895)^2 \approx 34$$

And we can write the general formula:

$$(\sqrt{x})^2 = x$$

Squaring Undoes
Square-Rooting

We assume $x \geq 0$, lest we take the square root of a negative number, resulting in a non-real number.

Another skill we’ll need for this chapter is the ability to simplify an expression like $(5\sqrt{3})^2$. Recall one of our five laws of exponents:

$$(ab)^2 = a^2b^2.$$

This rule means that we can simplify $(5\sqrt{3})^2$ as follows:

$$(5\sqrt{3})^2 = 5^2(\sqrt{3})^2 = 25 \cdot 3 = \mathbf{75}$$

Homework

1. While it’s the case that $(ab)^2 = a^2b^2$, is this rule true for addition? That is, does $(a + b)^2$ always equals $a^2 + b^2$?

2. Simplify each expression:

a. $(\sqrt{7})^2$ b. $(\sqrt{29})^2$ c. $(\sqrt{233})^2$ d. $(\sqrt{101})^2$
 e. $(2\sqrt{3})^2$ f. $(5\sqrt{5})^2$ g. $(9\sqrt{2})^2$ h. $(12\sqrt{7})^2$

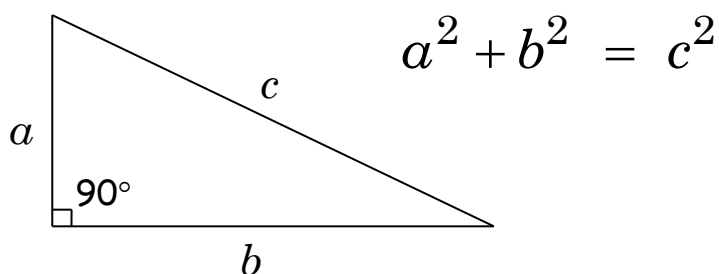
3. Example: $\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$

Simplify each radical:

a. $\sqrt{144}$ b. $\sqrt{225}$ c. $\sqrt{1}$ d. $\sqrt{0}$
 e. $\sqrt{-49}$ f. $\sqrt{14}$ g. $\sqrt{200}$ h. $\sqrt{45}$
 i. $\sqrt{98}$ j. $\sqrt{841}$ k. $\sqrt{48}$ l. $\sqrt{112}$

□ SOLVING RIGHT TRIANGLES

Recall the Pythagorean Theorem:



EXAMPLE 1: The legs are 6 and 10. Find the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow 6^2 + 10^2 = c^2 \Rightarrow 36 + 100 = c^2 \\ \Rightarrow c^2 &= 136 \Rightarrow c = \sqrt{136} = \sqrt{4 \cdot 34} = 2\sqrt{34} \end{aligned}$$

Notice that $c^2 = 136$ is a *quadratic equation*, and therefore probably has two solutions; in fact, the solutions are $\pm\sqrt{136}$, or $\pm 2\sqrt{34}$. But in this problem, we're talking about the hypotenuse of a triangle, whose length must be positive. So we immediately discard the negative solution and retain the positive one.

EXAMPLE 2: One leg is 20 and the hypotenuse is 30. Find the other leg.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow 20^2 + b^2 = 30^2 \\ \Rightarrow 400 + b^2 &= 900 \Rightarrow b^2 = 500 \\ \Rightarrow b &= \sqrt{500} = \sqrt{100 \cdot 5} = 10\sqrt{5} \end{aligned}$$

EXAMPLE 3: The legs are $2\sqrt{7}$ and $3\sqrt{13}$. Find the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow (2\sqrt{7})^2 + (3\sqrt{13})^2 = c^2 \\ \Rightarrow 28 + 117 &= c^2 \Rightarrow c^2 = 145 \Rightarrow c = \sqrt{145} \end{aligned}$$

EXAMPLE 4: The hypotenuse is $\sqrt{72}$ and one of the legs is 8. Find the other leg.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \Rightarrow a^2 + 8^2 &= (\sqrt{72})^2 \\ \Rightarrow a^2 + 64 &= 72 \\ \Rightarrow a^2 &= 8 \\ \Rightarrow a &= \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2} \end{aligned}$$

EXAMPLE 5: One of the legs is $3\sqrt{2}$ and the hypotenuse is $\sqrt{42}$. Find the other leg.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow a^2 + (3\sqrt{2})^2 = (\sqrt{42})^2 \\ &\Rightarrow a^2 + 18 = 42 \\ &\Rightarrow a^2 = 24 \\ &\Rightarrow a = \sqrt{24} = 2\sqrt{6} \end{aligned}$$

Homework

Calculate the EXACT answer to each Pythagorean problem:

4. The legs of a right triangle are 8 and 12. Find the hypotenuse.
5. The legs of a right triangle are 6 and 9. Find the hypotenuse.
6. Find the hypotenuse of a right triangle given that its legs have lengths of 5 and 7.
7. One leg of a right triangle is 15 and the hypotenuse is 20. Find the other leg.
8. The hypotenuse of a right triangle is 18 and one of its legs is 10. Find the other leg.
9. One leg of a right triangle is 5 and the hypotenuse is $\sqrt{43}$. Find the other leg.
10. Find the missing leg of a right triangle if the hypotenuse is $\sqrt{18}$ and one of its legs is 3.
11. The legs of a right triangle are $2\sqrt{3}$ and $3\sqrt{2}$. Find the hypotenuse.
12. The hypotenuse of a right triangle is $5\sqrt{7}$ and one of its legs is $2\sqrt{3}$. Find the length of the other leg.

13. A leg of a right triangle is $2\sqrt{8}$ and its hypotenuse is 10. Find the length of the other leg.
14. The leg of a right triangle is 5 and its hypotenuse is 4. Find the other leg.

Review Problems

15. In each problem, two of the sides of a right triangle are given (l = leg and h = hypotenuse). Find the third side.
- | | |
|-----------------------------------|-----------------------------------|
| a. $l = 7; l = 10$ | b. $l = 1; l = 3$ |
| c. $l = 4; l = 6$ | d. $l = \sqrt{8}; l = 2\sqrt{2}$ |
| e. $l = 4; l = \sqrt{10}$ | f. $l = \sqrt{2}; l = \sqrt{14}$ |
| g. $l = 5; h = 12$ | h. $l = \sqrt{5}; h = 2\sqrt{3}$ |
| i. $l = 6; h = 5\sqrt{2}$ | j. $l = \sqrt{7}; h = \sqrt{31}$ |
| k. $l = \sqrt{2}; h = 2\sqrt{5}$ | l. $l = \sqrt{5}; h = 4\sqrt{5}$ |
| m. $l = \sqrt{2}; l = \sqrt{7}$ | n. $l = 7; h = 8$ |
| o. $l = \sqrt{5}; h = \sqrt{13}$ | p. $l = 5; h = \sqrt{26}$ |
| q. $l = \sqrt{10}; h = \sqrt{41}$ | r. $l = \sqrt{11}; h = 8$ |
| s. $l = \sqrt{13}; l = \sqrt{13}$ | t. $l = 5; l = \sqrt{6}$ |
| u. $l = 4; l = 5$ | v. $l = 1; h = 2$ |
| w. $l = 1; l = 1$ | x. $l = 2\sqrt{2}; h = 2\sqrt{5}$ |
| y. $l = \sqrt{5}; h = \sqrt{21}$ | z. $l = 3\sqrt{2}; h = 20$ |

Solutions

- 1.** NO! You should know quite well that

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = \underline{a^2 + 2ab + b^2}$$

- 2.** a. 7 b. 29 c. 233 d. 101
e. 12 f. 125 g. 162 h. 1008

- 3.** a. 12 b. 15 c. 1 d. 0
e. Not a Real Number f. $\sqrt{14}$ g. $10\sqrt{2}$ h. $3\sqrt{5}$
i. $7\sqrt{2}$ j. 29 k. $4\sqrt{3}$ l. $4\sqrt{7}$

- 4.** $4\sqrt{13}$ **5.** $3\sqrt{13}$ **6.** $\sqrt{74}$ **7.** $5\sqrt{7}$ **8.** $4\sqrt{14}$ **9.** $3\sqrt{2}$

- 10.** 3 **11.** $\sqrt{30}$ **12.** $\sqrt{163}$ **13.** $2\sqrt{17}$

- 14.** This scenario is impossible, since the hypotenuse of a right triangle must be longer than either of its legs. Here's a mathematical proof:

$$a^2 + b^2 = c^2$$

$$\Rightarrow a^2 + 5^2 = 4^2 \quad (\text{presuming that the leg is 5 and the hypotenuse is 4})$$

$$\Rightarrow a^2 + 25 = 16$$

$$\Rightarrow a^2 = -9 \quad (\text{subtract 25 from each side of the equation})$$

$$\Rightarrow a = \sqrt{-9}, \quad \text{which is not a number in this class. And even if it becomes a number in a later class, it still couldn't be considered the length of a side of a triangle.}$$

- 15.** a. $\sqrt{149}$ b. $\sqrt{10}$ c. $2\sqrt{13}$ d. 4 e. $\sqrt{26}$
f. 4 g. $\sqrt{119}$ h. $\sqrt{7}$ i. $\sqrt{14}$ j. $2\sqrt{6}$
k. $3\sqrt{2}$ l. $5\sqrt{3}$ m. 3 n. $\sqrt{15}$ o. $2\sqrt{2}$
p. 1 q. $\sqrt{31}$ r. $\sqrt{53}$ s. $\sqrt{26}$ t. $\sqrt{31}$
u. 3 v. $\sqrt{3}$ w. $\sqrt{2}$ x. $2\sqrt{3}$ y. 4
z. $\sqrt{382}$

*“He who opens a school
door, closes a prison.”*

— Victor Hugo